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LETTER TO THE EDITOR

The deduction of the Lax representation for constrained flows from the adjoint representation

Yunbo Zeng and Yishen Li

Department of Mathematics, University of Science and Technology of China, Hefei 230026, People's Republic of China

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Abstract. For x- and t_n-finite-dimensional Hamiltonian systems obtained from the decompositions of zero-curvature equations, it is shown that their Lax representations can be deduced directly from the adjoint representations of the auxiliary linear problems. As a consequence, the zero-curvature representation for soliton hierarchy with source is presented.

It is well known that the Lax representation has played an important role in our understanding of complete integrability in classical mechanics and soliton theory [1–2]. Recently, systematic methods were developed to decompose each equation in a hierarchy of soliton equations into two commuting x- and t_n-integrable finite-dimensional Hamiltonian systems (FDHS) (see, for example, [3–9]). An important problem is to find the Lax representation for all these x- and t_n-FDHSs. By using the Gel'fand–Dikii approach, the Lax representation for x-constrained flows of Gel'fand–Dikii hierarchies was constructed in [9]. In this letter, within the framework of the zero-curvature representation theory, we will show that there is a natural way to reduce the adjoint representation of auxiliary linear problems to the Lax representation for all x- and t_n-FDHSs. As a consequence, the zero-curvature representation for the system consisting of the evolutions of eigenfunctions and zero-curvature equation with source is presented. This method can be applied to all hierarchies of zero-curvature equations as can be seen below. This indicates that all constrained flows of zero-curvature equations admit Lax representation.

As a model example, we consider the Jaulent–Miodek (JM) spectral problem [10] which can be written as

$$\varphi_x = M(u,\lambda)\varphi \equiv \begin{bmatrix} 0 & 1 \\ -\lambda^2+q\lambda+r & 0 \end{bmatrix}\varphi \qquad \varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \qquad u = \begin{bmatrix} q \\ r \end{bmatrix}. \tag{1}$$

The adjoint representation [2, 11] of (1) reads

$$V_x = [M, V] \equiv MV - VM \tag{2}$$

set

$$V = \sum_{m=0}^{\infty} V_m \lambda^{-m} \qquad V_m = \begin{bmatrix} a_m & b_m \\ c_m & -a_m \end{bmatrix}. \tag{3}$$

Equation (2) yields [11]

$$a_0=a_1=a_2=b_0=b_1=0 \qquad b_2=-1,\ b_3=-\tfrac{1}{2}q \qquad c_0=1 \qquad c_1=-\tfrac{1}{2}q$$

$$\begin{bmatrix} b_{m+2} \\ b_{m+1} \end{bmatrix} = L \begin{bmatrix} b_{m+1} \\ b_m \end{bmatrix} \qquad m=1,2,\ldots \tag{4a}$$

$$a_m=-\tfrac{1}{2}b_{m,x} \qquad c_m=a_{m,x}-b_{m+2}+qb_{m+1}+rb_m \tag{4b}$$

$$L=\begin{bmatrix} q-\frac{1}{2}D^{-1}q_x & r-\frac{1}{2}D^{-1}r_x-\frac{1}{4}D^2 \\ 1 & 0 \end{bmatrix} \qquad D=\frac{\partial}{\partial x} \qquad D^{-1}D=DD^{-1}=1.$$

Take [11]

$$N^{(n)}(u,\lambda)=\sum_{m=0}^{n} V_m\lambda^{n-m}+\Delta_n$$

$$\Delta_n(u,\lambda)=\begin{bmatrix} 0 & 0 \\ \lambda b_{n+1}+b_{n+2}-qb_{n+1} & 0 \end{bmatrix} \tag{5}$$

and let

$$\varphi_{t_n}=N^{(n)}(u,\lambda)\varphi=\left(\sum_{m=0}^{n} V_m\lambda^{n-m}+\Delta_n\right)\varphi. \tag{6}$$

Then the compatibility condition of (1) and (6) gives rise to the zero-curvature equation [1, 2]

$$M_{t_n}-N_x^{(n)}+[M,N^{(n)}]=0. \tag{7}$$

In fact, we have

$$M_{t_n}-N_x^{(n)}+[M,N^{(n)}]$$
$$=\begin{bmatrix} 0 & 0 \\ \lambda(q_{t_n}-2b_{n+1,x})+r_{t_n}-2b_{n+2,x}+b_{n+1}q_x+2qb_{n+1,x} & 0 \end{bmatrix} \tag{8}$$

which leads to the JM hierarchy [10, 11]

$$u_{t_n}=\begin{bmatrix} q \\ r \end{bmatrix}_{t_n}=J\begin{bmatrix} b_{n+2} \\ b_{n+1} \end{bmatrix}=J\frac{\delta H_{n+1}}{\delta u}$$
$$J=\begin{bmatrix} 0 & 2D \\ 2D & -q_x-2qD \end{bmatrix} \qquad H_1=-q \qquad H_n=\frac{1}{n-1}(2b_{n+2}-qb_{n+1}). \tag{9}$$

Furthermore, it is known [2, 7] that V determined by (2) also satisfies

$$V_{t_n}=[N^{(n)},V] \qquad n=1,2,\ldots \tag{10}$$

which is the adjoint representation of (6) and intimately connected with the fact that all flows (9) are commuting and integrable.

For N distinct λ_j, consider the system following from (1)

$$\begin{bmatrix} \psi_{1j} \\ \psi_{2j} \end{bmatrix}_x = M(u,\lambda_j)\begin{bmatrix} \psi_{1j} \\ \psi_{2j} \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -\lambda_j^2+q\lambda_j+r & 0 \end{bmatrix}\begin{bmatrix} \psi_{1j} \\ \psi_{2j} \end{bmatrix} \qquad j=1,\ldots,N. \tag{11}$$

A direct calculation gives

$$\frac{\delta\lambda_j}{\delta u}=\frac{1}{2}\begin{bmatrix}\lambda_j\psi_{1j}^2\\ \psi_{1j}^2\end{bmatrix} \qquad L\begin{bmatrix}\lambda_j\psi_{1j}^2\\ \psi_{1j}^2\end{bmatrix}=\lambda_j\begin{bmatrix}\lambda_j\psi_{1j}^2\\ \psi_{1j}^2\end{bmatrix}. \tag{12}$$

It was pointed out in [4-7] that the following equation

$$\frac{\delta H_{k+1}}{\delta u}+\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}=0 \tag{13}$$

determines a finite-dimensional invariant set for the flows (9). This property ensures that (9), (11) and (13) are consistent. Equation (13) together with (12) yields

$$\frac{\delta H_{k+1}}{\delta u}=\begin{bmatrix}b_{k+2}\\ b_{k+1}\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}\langle\Lambda\Psi_1,\Psi_1\rangle\\ \langle\Psi_1,\Psi_1\rangle\end{bmatrix}. \tag{14}$$

Hereafter $\Psi_i=(\psi_{i1},\ldots,\psi_{iN})^T$, $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$ and $\langle\cdot,\cdot\rangle$ denotes inner product in R^N. As proposed in [6, 7], we consider a system consisting of (11) and (14) for any fixed k:

$$\Psi_{1x}=\Psi_2 \qquad \Psi_{2x}=-\Lambda_2\Psi_1+q\Lambda\Psi_1+r\Psi_1 \tag{15a}$$

$$\frac{\delta H_{k+1}}{\delta u}=\begin{bmatrix}b_{k+2}\\ b_{k+1}\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}\langle\Lambda\Psi_1,\Psi_1\rangle\\ \langle\Psi_1,\Psi_1\rangle\end{bmatrix}. \tag{15b}$$

Obviously, (15a, b) is a Euler-Lagrange system

$$\frac{\delta\pounds}{\delta\Psi_2}=0 \qquad \frac{\delta\pounds}{\delta\Psi_1}=0 \qquad \frac{\delta\pounds}{\delta u}=0$$

$$\pounds=H_{k+1}-\langle\Psi_2,\Psi_2\rangle-\langle\Lambda^2\Psi_1,\Psi_1\rangle+\tfrac{1}{2}q\langle\Lambda\Psi_1,\Psi_1\rangle+\tfrac{1}{2}r\langle\Psi_1,\Psi_1\rangle+\langle\Psi_{1x},\Psi_2\rangle.$$

Thus, for even k, by introducing so-called Jacobi-Ostrogradsky coordinates [12], (15a, b) can be transformed to a canonical FDHS. Also for the system following from (6) and (9)

$$\begin{bmatrix}\psi_{1j}\\ \psi_{2j}\end{bmatrix}_{t_n}=N^{(n)}(u,\lambda_j)\begin{bmatrix}\psi_{1j}\\ \psi_{2j}\end{bmatrix} \qquad j=1,\ldots,N \tag{16a}$$

$$\begin{bmatrix}q\\ r\end{bmatrix}_{t_n}=J\begin{bmatrix}b_{n+2}\\ b_{n+1}\end{bmatrix} \tag{16b}$$

under (15a, b) and the Jacobi-Ostrogradsky coordinates, (16a, b) can be transformed to a canonical FDHS with t_n as independent variable (see examples at the end of the letter). These two FDHSs are commuting and integrable, and called x-part and t_n-part of the decomposition of (9), respectively. As we emphasized in [6, 7], this decomposition provides a method of separation of variables for solving equation (9).

From (4a), (12) and (14), one finds

$$b_{m+1}=-\tfrac{1}{2}\langle\Lambda^{m-k}\Psi_1,\Psi_1\rangle \qquad m\geqslant k \tag{17a}$$

which together with (4b) and (15a) leads to

$$a_{m+1}=\tfrac{1}{2}\langle\Lambda^{m-k}\Psi_1,\Psi_2\rangle \qquad c_{m+1}=\tfrac{1}{2}\langle\Lambda^{m-k}\Psi_2,\Psi_2\rangle \qquad m\geqslant k. \tag{17b}$$

Equation (17) results in the following kind of formulae:

$$\lambda^k\sum_{m=k}^{\infty}b_{m+1}\lambda^{-m-1}=-\frac{1}{2}\frac{1}{\lambda}\sum_{m=0}^{\infty}\sum_{j=1}^{N}\left(\frac{\lambda_j}{\lambda}\right)^m\psi_{1j}^2=-\frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{1j}^2}{\lambda-\lambda_j}. \tag{18}$$

Hence, notice (5), $\lambda^k V$ under (15) can be written as

$$U^{(k)} = \lambda^k V = N^{(k)} + N_0 \tag{19a}$$

$$N_0 = \frac{1}{2}\sum_{j=1}^{N} \frac{1}{\lambda-\lambda_j}\begin{bmatrix} \psi_{1j}\psi_{2j} & -\psi_{1j}^2 \\ \psi_{2j}^2 & -\psi_{1j}\psi_{2j} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{2}\lambda\langle\Psi_1, \Psi_1\rangle + \frac{1}{2}\langle\Lambda\Psi_1, \Psi_1\rangle - \frac{1}{2}q\langle\Psi_1, \Psi_1\rangle & 0 \end{bmatrix}. \tag{19b}$$

Then it follows from (2) that $U^{(k)}$ also satisfies

$$U_x^{(k)} = [M, U^{(k)}].$$

Conversely, we have:

Proposition 1. By substituting $U^{(k)}$, the adjoint representation (2) reduces to the Lax representation for (15):

$$U_x^{(k)} = [M, U^{(k)}] \tag{20}$$

with the Lax pair defined by

$$\varphi_x = M\varphi \qquad U^{(k)}\varphi = \mu\varphi. \tag{21}$$

Proof. Taking $M_{t_k} = 0$, equation (8) for $n = k$ implies that

$$N_x^{(k)} - [M, N^{(k)}] = \begin{bmatrix} 0 & 0 \\ 2b_{k+1,x}\lambda + 2b_{k+2,x} - q_x b_{k+1} - 2qb_{k+1,x} & 0 \end{bmatrix}.$$

Then by a direct calculation of $(N_{0,x} - [M, N_0])$, it can be verified that (20) gives rise to (15).

As a consequence, notice (8), we immediately obtain:

Proposition 2. The system

$$\Psi_{1x} = \Psi_2 \qquad \Psi_{2x} = -\Lambda^2\Psi_1 + q\Lambda\Psi_1 + r\Psi_1 \tag{22a}$$

$$u_{t_k} = J\begin{bmatrix} b_{k+2} \\ b_{k+1} \end{bmatrix} + \frac{1}{2}J\begin{bmatrix} \langle\Lambda\Psi_1, \Psi_1\rangle \\ \langle\Psi_1, \Psi_1\rangle \end{bmatrix} \tag{22b}$$

admits a zero-curvature representation

$$M_{t_k} - U_x^{(k)} + [M, U^{(k)}] = 0 \tag{23}$$

with the auxiliary linear problems given by

$$\varphi_x = M\varphi \qquad \varphi_{t_k} = U^{(k)}\varphi. \tag{24}$$

It is obvious that (15) is just the stationary equation of (22) ($u_{t_k} = 0$). Similarly, it can be shown that:

Proposition 3. By using $U^{(k)}$ and the expression of $N^{(n)}$ under (15), the adjoint representation (10) reduces to the Lax representation for (16) under (15):

$$U_{t_n}^{(k)} = [N^{(n)}, U^{(k)}] \tag{25}$$

with

$$\varphi_{t_n} = N^{(n)}\varphi \qquad U^{(k)}\varphi = \mu\varphi. \tag{26}$$

Finally it is easy to see that $\mathrm{tr}(U^{(k)})^2$ is the generating function of the integrals of motion for (15) and (16).

To illustrate the above propositions, we present some examples below.

(i) When $k=2$, (14) reads

$$q=\langle\Psi_1,\Psi_1\rangle \qquad r=\langle\Lambda\Psi_1,\Psi_1\rangle-\tfrac{3}{4}\langle\Psi_1,\Psi_1\rangle^2 \tag{27}$$

and (15) becomes

$$\begin{gathered}\Psi_{1x}=\frac{\partial\tilde{H}_0}{\partial\Psi_2} \qquad \Psi_{2x}=-\frac{\partial\tilde{H}_0}{\partial\Psi_1}\\ \tilde{H}_0=\tfrac{1}{2}\langle\Psi_2,\Psi_2\rangle+\tfrac{1}{2}\langle\Lambda^2\Psi_1,\Psi_1\rangle-\tfrac{1}{2}\langle\Psi_1,\Psi_1\rangle\langle\Lambda\Psi_1,\Psi_1\rangle+\tfrac{1}{8}\langle\Psi_1,\Psi_1\rangle^3\end{gathered} \tag{28}$$

the Lax representation for (28) is given by (20) with

$$\begin{gathered}M=\begin{bmatrix}0 & 1\\ -\lambda^2+\langle\Psi_1,\Psi_1\rangle\lambda+\langle\Lambda\Psi_1,\Psi_1\rangle-\tfrac{3}{4}\langle\Psi_1,\Psi_1\rangle^2 & 0\end{bmatrix}\\ U^{(2)}=\begin{bmatrix}0 & -1\\ \lambda^2-\langle\Psi_1,\Psi_1\rangle\lambda-\langle\Lambda\Psi_1,\Psi_1\rangle+\tfrac{3}{4}\langle\Psi_1,\Psi_1\rangle^2 & 0\end{bmatrix}+N_0\end{gathered} \tag{29}$$

the case for $n=2$ is trivial, for $n=3$, (16) under (27) and (28) becomes

$$\begin{aligned}&\Psi_{1t_3}=\frac{\partial\tilde{H}_3}{\partial\Psi_2} \qquad \Psi_{2t_3}=-\frac{\partial\tilde{H}_3}{\partial\Psi_1}\\ &\tilde{H}_3=-\tfrac{1}{2}\langle\Lambda^3\Psi_1\Psi_1\rangle-\tfrac{1}{2}\langle\Psi_2,\Psi_2\rangle+\tfrac{1}{4}\langle\Psi_1,\Psi_2\rangle^2-\tfrac{1}{4}\langle\Psi_1,\Psi_1\rangle\langle\Psi_2,\Psi_2\rangle\\ &\qquad+\tfrac{1}{4}\langle\Psi_1,\Psi_1\rangle\langle\Lambda^2\Psi_1,\Psi_1\rangle-\tfrac{1}{8}\langle\Psi_1,\Psi_1\rangle^2\langle\Lambda\Psi_1,\Psi_1\rangle+\tfrac{1}{4}\langle\Lambda\Psi_1,\Psi_1\rangle^2\end{aligned} \tag{30}$$

the Lax representation for (30) is (25) with $U^{(2)}$ given by (29) and

$$N^{(3)}=\begin{bmatrix}\tfrac{1}{2}\langle\Psi_1,\Psi_2\rangle & -\lambda-\tfrac{1}{2}\langle\Psi_1,\Psi_1\rangle\\ A_1 & -\tfrac{1}{2}\langle\Psi_1,\Psi_2\rangle\end{bmatrix}$$

$$\begin{aligned}A_1=\lambda^3-\tfrac{1}{2}\langle\Psi_1,\Psi_1\rangle\lambda^2+(\tfrac{1}{4}\langle\Psi_1,\Psi_1\rangle^2-\langle\Lambda\Psi_1,\Psi_1\rangle)\lambda+\tfrac{1}{2}\langle\Psi_2,\Psi_2\rangle\\ -\tfrac{1}{2}\langle\Lambda^2\Psi_1,\Psi_1\rangle+\tfrac{1}{2}\langle\Psi_1,\Psi_1\rangle\langle\Lambda\Psi_1,\Psi_1\rangle.\end{aligned}$$

(ii) When $k=4$, (14) gives

$$\begin{gathered}\frac{\delta H_5}{\delta u}=\begin{bmatrix}b_6\\ b_5\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}\langle\Lambda\Psi_1,\Psi_1\rangle\\ \langle\Psi_1,\Psi_1\rangle\end{bmatrix}\\ H_5=-\tfrac{7}{128}q^5-\tfrac{5}{16}q^3r-\tfrac{5}{32}qq_x^2-\tfrac{3}{8}qr^2-\tfrac{1}{8}q_xr_x\end{gathered} \tag{31}$$

by introducing the Jacobi–Ostrogradsky coordinates

$$\begin{gathered}Q\equiv[\psi_{11},\ldots,\psi_{1N},q_1,q_2]^T \qquad P\equiv[\psi_{21},\ldots,\psi_{2N},p_1,p_2]^T\\ q_1=q,\ q_2=r \qquad p_1=-\tfrac{5}{16}qq_x-\tfrac{1}{8}r_x \qquad p_2=-\tfrac{1}{8}q_x\end{gathered} \tag{32}$$

(15) can be written as

$$\begin{aligned}&Q_x=\frac{\partial\tilde{H}_0}{\partial P} \qquad P_x=-\frac{\partial\tilde{H}_0}{\partial Q}\\ &\tilde{H}_0=\tfrac{1}{2}\langle\Lambda^2\Psi_1,\Psi_1\rangle-\tfrac{1}{2}q_1\langle\Lambda\Psi_1,\Psi_1\rangle-\tfrac{1}{2}q_2\langle\Psi_1,\Psi_1\rangle+\tfrac{1}{2}\langle\Psi_2,\Psi_2\rangle\\ &\qquad-8p_1p_2+\tfrac{7}{128}q_1^5+\tfrac{5}{16}q_1^3q_2+10q_1p_2^2+\tfrac{3}{8}q_1q_2^2.\end{aligned} \tag{33}$$

The Lax representation for (32) is given by (20) with

$$M=\begin{bmatrix} 0 & 1 \\ -\lambda^2+q_1\lambda+q_2 & 0 \end{bmatrix}$$

$$U^{(4)}=\begin{bmatrix} -2p_2\lambda-2p_1+2q_1p_2 & -\lambda^2-\frac{1}{2}q_1\lambda-\frac{3}{8}q_1^2-\frac{1}{2}q_2 \\ A_2 & 2p_2\lambda+2p_1-2q_1p_2 \end{bmatrix}+N_0 \tag{34}$$

$$\begin{aligned} A_2=\lambda^4-\tfrac{1}{2}q_1\lambda^3-\tfrac{1}{8}(4q_2+q_1^2)\lambda^2+\tfrac{1}{4}(q_1^3+2q_1q_2-4\langle\Psi_1,\Psi_1\rangle)\lambda \\ -\tfrac{5}{64}q_1^4+4p_2^2+\tfrac{1}{4}q_2^2-\langle\Lambda\Psi_1,\Psi_1\rangle+q_1\langle\Psi_1,\Psi_1\rangle \end{aligned}$$

for $n=3$, (16) under (32) and (33) can be transformed to

$$Q_{t_3}=\frac{\partial\tilde{H}_3}{\partial P} \qquad P_{t_3}=-\frac{\partial\tilde{H}_3}{\partial Q}$$

$$\begin{aligned} \tilde{H}_3=-\tfrac{1}{2}\langle\Lambda^3\Psi_1,\Psi_1\rangle-\tfrac{1}{2}\langle\Lambda\Psi_2,\Psi_2\rangle+\tfrac{1}{4}\langle\Psi_1,\Psi_1\rangle^2+\tfrac{1}{4}\langle\Lambda^2\Psi_1,\Psi_1\rangle q_1+\tfrac{1}{4}(q_1^2+2q_2)\langle\Lambda\Psi_1,\Psi_1\rangle \\ -2p_2\langle\Psi_1,\Psi_2\rangle-\tfrac{1}{4}q_1\langle\Psi_2,\Psi_2\rangle-\tfrac{1}{16}(5q_1^3+8q_1q_2)\langle\Psi_1,\Psi_1\rangle \\ -\tfrac{1}{8}q_2^3-2p_2^2q_2+\tfrac{15}{512}q_1^6-\tfrac{3}{32}q_1^2q_2^2+\tfrac{5}{128}q_1^4q_2+4p_1^2-8q_1p_1p_2+\tfrac{5}{2}q_1^2p_2^2. \end{aligned} \tag{35}$$

The Lax representation for (35) is (25) with $U^{(4)}$ given by (34) and

$$N^{(3)}=\begin{bmatrix} -2p_2 & -\lambda-\frac{1}{2}q_1 \\ \lambda^3-\frac{1}{2}q_1\lambda^2-\frac{1}{2}(q_1^2+2q_2)\lambda+\frac{5}{8}q_1^3+q_1q_2-\langle\Psi_1,\Psi_1\rangle & 2p_2 \end{bmatrix}.$$

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